
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Trial Examination

FORM VI

MATHEMATICS EXTENSION 2

Tuesday 9th August 2016

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 72 boys

Examiner

DNW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The value of i^{2016} is:

(A) i

(B) -1

(C) $-i$

(D) 1

QUESTION TWO

At time t a particle is at position x and travelling with velocity v. Which of the following expressions is NOT equal to the acceleration of the particle?

(A) $\dfrac{d^2x}{dt^2}$

(B) $\dfrac{dv}{dt}$

(C) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

(D) $\left(\dfrac{dx}{dt}\right)^2$

QUESTION THREE

Which expression is equal to $\displaystyle\int \frac{1}{x^2+2x+3}\,dx$?

(A) $\tan^{-1}\left(\dfrac{x+1}{2}\right)+C$

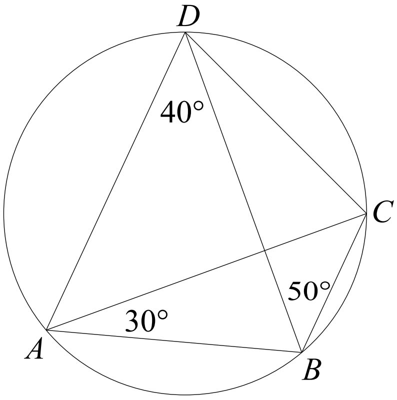
(B) $\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)+C$

(C) $\dfrac{1}{2}\times\tan^{-1}\left(\dfrac{x+1}{2}\right)+C$

(D) $\dfrac{1}{\sqrt{2}}\times\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)+C$

Examination continues next page ...

QUESTION FOUR



The sketch above shows cyclic quadrilateral $ABCD$ with $\angle BAC = 30°$, $\angle ADB = 40°$ and $\angle CBD = 50°$. From this information, the size of $\angle ABD$ is:

(A) $20°$

(B) $40°$

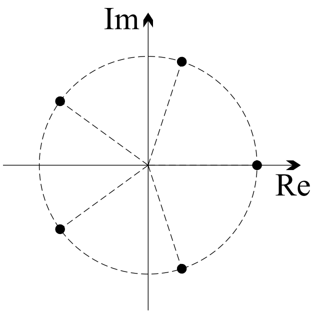
(C) $60°$

(D) $65°$

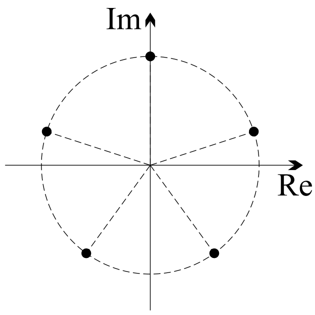
QUESTION FIVE

Which of the following Argand diagrams shows the solutions of $z^5 - i = 0$?

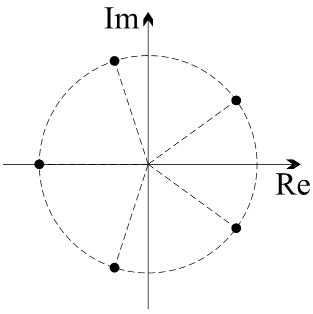
(A)



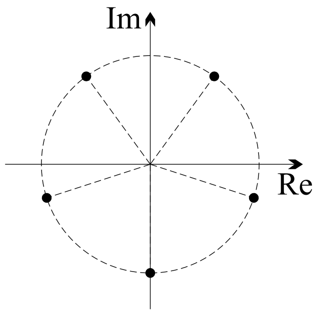
(B)



(C)



(D)



Examination continues overleaf ...

QUESTION SIX

A certain function $f(x)$ has the following properties: $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 3$.
Which of the following is possible for all values of x?

(A) $f''(x) > 0$ and $f'(x) > 0$

(B) $f''(x) > 0$ and $f'(x) < 0$

(C) $f''(x) < 0$ and $f'(x) > 0$

(D) $f''(x) < 0$ and $f'(x) < 0$

QUESTION SEVEN

The polynomial $P(z)$ has real coefficients and $P(0) = -1$. The imaginary number α and the real number β satisfy:

$$P(\alpha) = 0\,, \quad P(\beta) = 0 \quad \text{and} \quad P'(\beta) = 0\,.$$

The degree of $P(z)$ is at least:

(A) 2

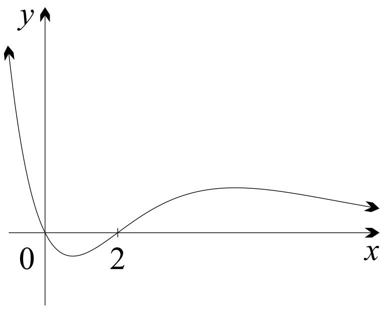
(B) 3

(C) 4

(D) 5

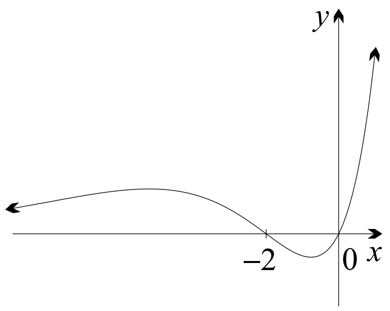
Examination continues next page ...

QUESTION EIGHT

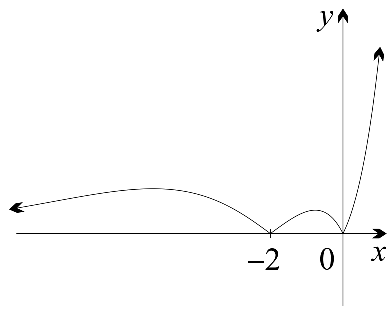


Above is the graph of $y = f(x)$. The correct graph of $y = |f(-x)|$ is:

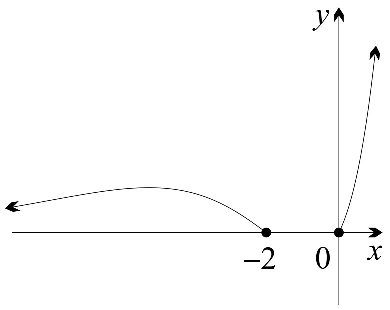
(A)



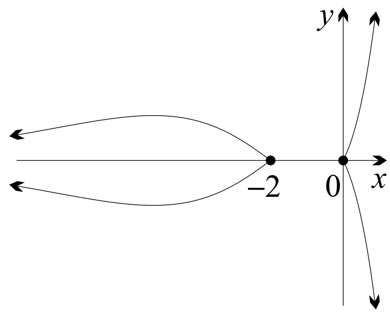
(B)



(C)



(D)



QUESTION NINE

Let $z = \cos\theta + i\sin\theta$, where θ is acute. The value of $\arg(z^2 + 1)$ is:

(A) $\dfrac{\theta}{2}$

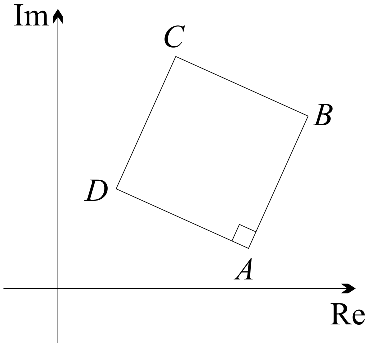
(B) θ

(C) 2θ

(D) 4θ

Examination continues overleaf ...

QUESTION TEN



The Argand diagram above shows the square $ABCD$ in the first quadrant. The point A represents the complex number z and the point C represents w. Which of the following represents the point B?

(A) $\dfrac{z+w}{2}+\dfrac{i(z-w)}{2}$

(B) $\dfrac{z+w}{2}-\dfrac{i(z-w)}{2}$

(C) $\dfrac{z-w}{2}+\dfrac{i(z+w)}{2}$

(D) $\dfrac{z-w}{2}-\dfrac{i(z+w)}{2}$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Express $\dfrac{5-i}{2+i}$ in the form $x+iy$, where x and y are real. **2**

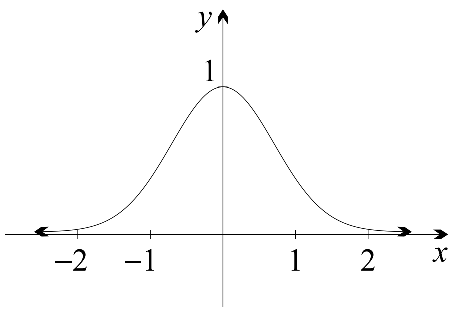
(b) Consider the region in the Argand diagram where $|z-2i| \le 1$ and $\text{Re}(z) \le 0$.

(i) Sketch the region. **2**

(ii) What range of values does $\arg(z)$ take? **2**

(c) Use the substitution $t = \tan(\frac{x}{2})$ to help evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\,dx$. **4**

(d) **2**



The graph of $y = e^{-x^2}$ is shown above. Draw a one-third page sketch of the graph of

$$y = (1-x^2)e^{-x^2},$$

without the aid of calculus. Show all intercepts.

(e) The region in the first quadrant below $y = 4x - x^3$ is rotated about the y-axis to form a solid. Use the method of cylindrical shells to find the volume of this solid. **3**

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) The equation $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$ represents a hyperbola. Find the eccentricity e. **1**

(b) Find the gradient of the tangent to the curve $x^2 - 2xy + 3y^2 = 11$ at the point $(2, -1)$. **3**

(c) Let $P(z) = z^4 + 2z^3 + 3z^2 + 4z + 2$.

(i) Show that $z = -1$ is a double root of $P(z) = 0$. **1**

(ii) The other two roots are complex. Use the sum and product of roots to find them. **3**

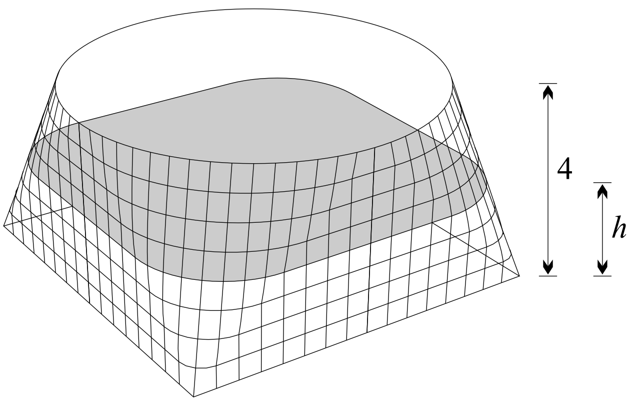
(d) A 1 kg object is moving along the x-axis and is subject to a resistive force of $\frac{1}{3}v^3 + v$, where v is its velocity in metres per second. That is, its equation of motion is **3**

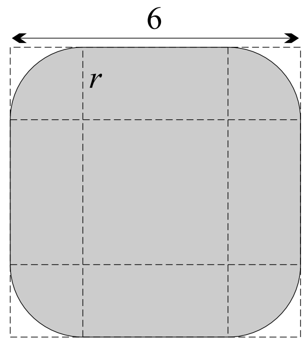
$$v\frac{dv}{dx} = -\tfrac{1}{3}v^3 - v\,.$$

Initially the object is at the origin, and its velocity is 3 m/s.

Find v as a function of x.

(e)





An artist has created a sculpture which has a square base with side length 6 cm and a circular top with radius 3 cm. The height of the solid is 4 cm. A cross-section at height h is shown shaded grey, and a separate top view of the cross-section is shown on the right. This cross-section is a square with the corners replaced by quadrants of radius r. The radius r varies in direct proportion with h. That is, $r = kh$ for some constant k.

(i) Show that the area A of the cross-section is given by **2**

$$A = 36 + \tfrac{9}{16}(\pi - 4)h^2.$$

(ii) Hence find the exact volume of the solid. **2**

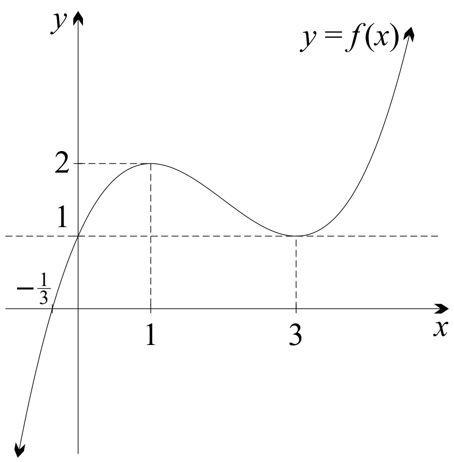
QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Suppose that $y = f(x)$ has a stationary point at $x = a$, and that both $f(a) \neq 0$ and $f''(a) \neq 0$. Let $g(x) = \dfrac{1}{f(x)}$.

(α) Prove that the graph of $y = g(x)$ also has a stationary point at $x = a$. **1**

(β) Show that at the stationary point the sign of $g''(a)$ is opposite to $f''(a)$. **2**

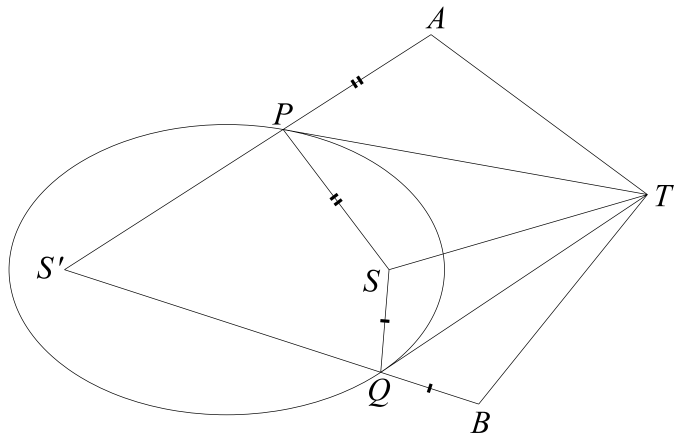
(ii) **2**



The graph of $y = f(x)$ is shown above. You may assume that both $f''(1) \neq 0$ and $f''(3) \neq 0$.

Use the results of part (i) to help sketch the graph of $y = \dfrac{1}{f(x)}$. Clearly show the behaviour at any turning points, and any other significant features.

(b)



The diagram above shows an ellipse with foci at S and S'. Let the length of the major axis be $2a$. The tangents from points P and Q on the ellipse meet at T. Extend $S'P$ to A with $PA = PS$, and extend $S'Q$ to B with $QB = QS$.

In this question you may assume the reflection property of the ellipse.

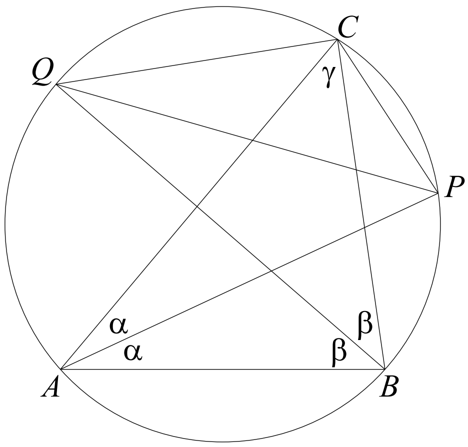
(i) Prove that $AT = ST$. **2**

(ii) Explain why $S'A = 2a$. **1**

(iii) Hence prove that $S'T$ bisects $\angle PS'Q$. **2**

Examination continues overleaf ...

(c)



The diagram above shows a circle with a chord AB of fixed length. A variable point C lies on the major arc AB. The angle bisector of $\angle BAC$ meets the circle again at P, and the angle bisector of $\angle ABC$ meets the circle at Q. Let $\angle BAP = \alpha$, $\angle ABQ = \beta$ and $\angle ACB = \gamma$.

Copy or trace the diagram into your writing booklet.

(i) Show that $\angle PCQ = \alpha + \beta + \gamma$. **2**

(ii) Hence prove that $\angle PCQ$ is constant, regardless of the location of point C. **2**

(iii) Give a reason why the length of PQ is constant. **1**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = \cos\theta + i\sin\theta$. Then, by de Moivre's theorem, it is known that **3**

$$z^n + z^{-n} = 2\cos n\theta \qquad \text{and} \qquad z^n - z^{-n} = 2i\sin n\theta\,.$$

Use these two results to show that

$$16\cos^3\theta\sin^2\theta = 2\cos\theta - \cos 3\theta - \cos 5\theta\,.$$

(b) (i) Solve $z^5 = 1$. **1**

(ii) Let α be the complex root in the first quadrant of the Argand diagram. Show that $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$. **1**

(iii) Find a quadratic equation which has roots $(\alpha^4 + \alpha)$ and $(\alpha^3 + \alpha^2)$. **2**

(iv) Solve this equation and hence evaluate $\cos\frac{2\pi}{5}$. **2**

(c) The polynomial $P(z) = z^4 - 2z^3 + 2z^2 - 10z + 25$ has two complex zeroes α and $i\,\alpha$.

(i) By considering the equations $P(z) = 0$ and $P(iz) = 0$, show that α is a root of **2**

$$(1+i)z^2 - 2z + 5(1-i) = 0\,.$$

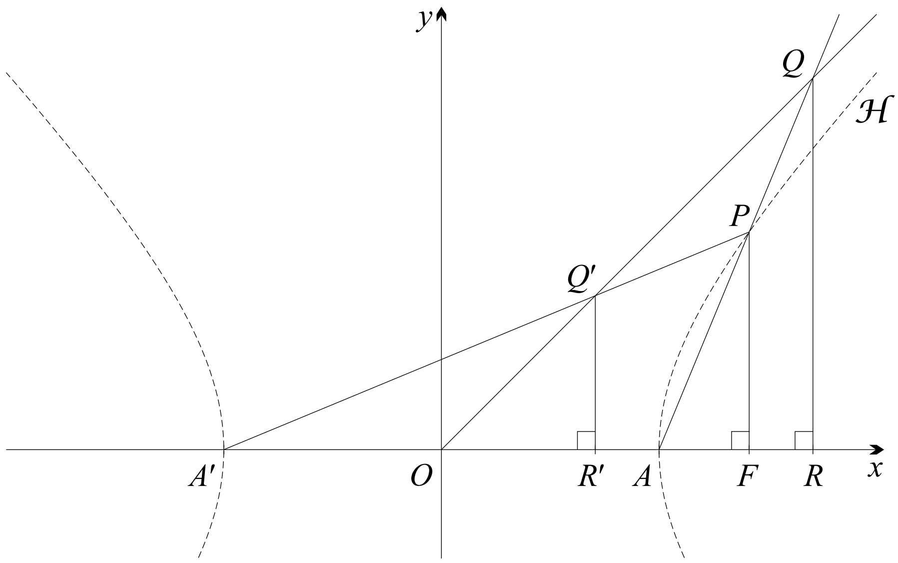
(ii) Solve the above quadratic and thus find all the zeroes of $P(z)$. **4**

Examination continues next page ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find the locus of z given that $\dfrac{z-1}{z+1}$ is imaginary. **3**

(b)



The diagram above shows the hyperbola $\mathcal{H}$ with vertices at $A(a, 0)$ and $A'(-a, 0)$, and centre $O(0, 0)$. The point $P(a\sec\theta, b\tan\theta)$ is in the first quadrant and on $\mathcal{H}$. Let F be the foot of the perpendicular from P to the x-axis. The asymptote $y = \frac{b}{a}x$ intersects AP at $Q(a\mu, b\mu)$ and intersects $A'P$ at $Q'(a\lambda, b\lambda)$. You may assume $\mu > \lambda$. The perpendiculars from Q and Q' meet the x-axis at R and R' respectively.

(i) Show that $Q'Q = (\mu - \lambda)\sqrt{a^2 + b^2}$. **1**

(ii) You may assume that $\triangle AFP \;|||\; \triangle ARQ$. Use the ratios of matching sides to express $\dfrac{\mu}{\mu - 1}$ in terms of θ alone. **1**

(iii) Find a similar ratio for λ. **1**

(iv) Multiply your results for parts (ii) and (iii) and hence show that the length of $Q'Q$ is constant. **2**

(c) By considering $(x - \frac{1}{x})^2$, show that $a + \frac{1}{a} \geq 2$ whenever $a > 0$. **1**

(d) A 1 kg mass is projected vertically upward from ground level with initial velocity V_0. As it moves it is influenced by gravity $g\,\text{m/s}^2$ and an air resistance equal to kv, where v is its velocity in metres per second and k is a positive constant. Let y be its height above ground level after t seconds, with upwards as positive. You may assume that the equation of motion is

$$\ddot{y} = -g - kv\,.$$

(i) Show that the velocity at time t is given by **3**

$$v(t) = \tfrac{g}{k}\Big((1 + \tfrac{k}{g}V_0)e^{-kt} - 1\Big)\,.$$

(ii) Show that the time T taken to reach the maximum height satisfies **1**

$$e^{kT} = 1 + \tfrac{k}{g}V_0\,.$$

(iii) From part (i), the velocity s seconds before it reaches the maximum height is **2**

$$v(T - s) = \tfrac{g}{k}\Big((1 + \tfrac{k}{g}V_0)e^{-k(T-s)} - 1\Big)\,.$$

It can be shown that $v(T + s)$ is the velocity s seconds after it reaches the maximum height, and that $v(T + s)$ is negative to indicate downwards velocity. (Do NOT show this.)

Use part (c) to help show that $v(T - s) + v(T + s) \geq 0$.

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Find the values of A, B and C such that **2**

$$\frac{1}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}\,.$$

(ii) Find and simplify $I(N) = \displaystyle\int_0^N \frac{dx}{x^3+x^2+x+1}\,.$ **3**

(iii) Hence evaluate $\displaystyle\int_0^\infty \frac{dx}{x^3+x^2+x+1}\,.$ **1**

(b) Let $I_n = \displaystyle\int_0^1 x^n e^{-x}\,dx$, where $n \geq 0$ is an integer.

(i) Use integration by parts to show that $I_n = \dfrac{1}{n+1}\Big(e^{-1} + I_{n+1}\Big)\,.$ **1**

(ii) Use induction to show that **3**

$$I_0 = e^{-1}\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) + \frac{1}{n!}\,I_n \qquad \text{for } n \geq 1\,.$$

(iii) You may assume that $\displaystyle\lim_{n\to\infty} I_n = 0$ and that the series in part (ii) converges to a limit. Use these assumptions to show that **1**

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

(iv) Suppose that e is rational. That is, suppose that $e = \dfrac{p}{q}$, where p and q are integers with $p > 0$ and $q \geq 2$. Use part (iii) to show that **1**

$$p(q-1)! = \left(\sum_{k=0}^{q} \frac{q!}{k!}\right) + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots$$

(v) Let $f = \dfrac{1}{q+1} + \dfrac{1}{(q+1)(q+2)} + \dfrac{1}{(q+1)(q+2)(q+3)} + \ldots$ **2**

By comparing f with a suitable geometric progression, show that $0 < f < \frac{1}{2}$.

(vi) Hence prove that e is in fact irrational. **1**

End of Section II

END OF EXAMINATION

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B L A N K   P A G E

Multiple Choice (with comments on errors)

Q 1 (D) $2016 = 4 \times 504 + 0$ so $i^{2016} = i^0 = 1$.
(A) $i^{2017} = i$ (B) $i^{2018} = -1$ (C) $i^{2019} = -i$

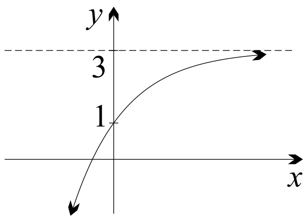
Q 2 (D) This is v^2, and $v^2 \neq a$.
(A) acceleration (B) acceleration (C) acceleration

Q 3 (D) $x^2 + 2x + 3 = (x+1)^2 + (\sqrt{2})^2$ then use the formula for $\tan^{-1}$, $a = \sqrt{2}$
(A) a wrong, wrong formula (B) wrong formula (C) a wrong

Q 4 (C) $\angle CAD = 50^\circ$ (angle in same segment)
$\angle DBA = 60^\circ$ (angle sum of $\triangle ABD$)
(A) $\angle BDC = 30^\circ$ but opposite angles of $ABCD$ are NOT equal.
(B) Incorrect angle sum of triangle
(D) The diagram is not to scale.

Q 5 (B) One solution must be $z = i$. The others are equally spaced from this.
(A) $z^5 - 1 = 0$ (C) $z^5 + 1 = 0$ (D) $z^5 + i = 0$

Q 6 (C) $f(0) < \lim\limits_{x\to\infty} f(x)$ so $f'(x) > 0$
thus $f'(x) > 0$ and $\lim\limits_{x\to\infty} f'(x) = 0$ so $f''(x) < 0$
(See the example graph on the right.)



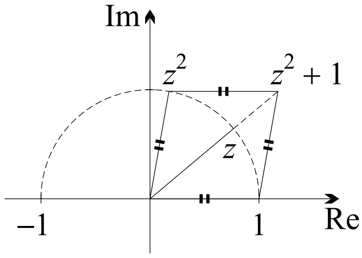
The other options are wrong because:

(A) $f''(x) > 0$ (B) $f''(x) > 0$ and $f'(x) < 0$ (D) $f'(x) < 0$

Q 7 (C) β is a double root, and for real coefficients complex roots come in pairs.
(A) not just α and β
(B) omitted $\overline{\alpha}$ or omitted β as double root
(D) $P(0) = -1$ does not imply an extra zero

Q 8 (B) reflect in y axis, $y = |f| > 0$, no restriction on x.
(A) $y < 0$ (C) restriction on x (D) restriction on x, $y < 0$

Q 9 (B) angles in a rhombus
Alternatively
$$\begin{aligned} z^2 + 1 &= (\cos 2\theta + 1) + i \sin 2\theta \\ &= 2\cos^2\theta + 2i \sin\theta \cos\theta \\ &= 2\cos\theta(\cos\theta + i\sin\theta)\,. \end{aligned}$$
(A) $\arg(z+1)$ (C) $\arg(z^2)$ (D) $\arg(z^4)$



Q 10 (A) Let M be the mid-point of AC then $\dfrac{z+w}{2} = \overrightarrow{OM}$ and $\dfrac{z-w}{2} = \frac{1}{2}\overrightarrow{CA}$.
So $\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} = \overrightarrow{OM} + i \times \frac{1}{2}\overrightarrow{CA}$

(B) point D (C) $i \times \overrightarrow{OD}$ (D) $-i \times \overrightarrow{OB}$

QUESTION ELEVEN (15 marks)

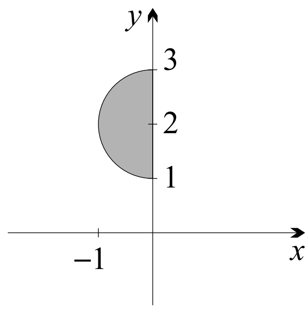
Marks

(a) Realise the denominator using its conjugate: **2**

$$\frac{5-i}{2+i} = \frac{5-i}{2+i} \times \frac{2-i}{2-i}$$ √

$$= \frac{9-7i}{5} \quad \text{or} \quad \tfrac{9}{5} - \tfrac{7}{5}i$$ √

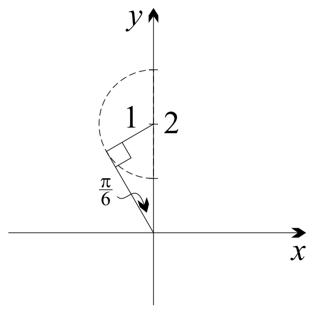
(b) (i) **2**



Inside and on the circle centre $2i$, radius 1 √

and in the left half-plane. √

(ii) **2**



$$\tfrac{\pi}{2} \le \arg(z) \le \tfrac{\pi}{2} + \tfrac{\pi}{6}$$ √

so $$\tfrac{\pi}{2} \le \arg(z) \le \tfrac{2\pi}{3}$$ √

(c) From $t = \tan\frac{\theta}{2}$ **4**

$$d\theta = \frac{2}{1+t^2}\,dt\,.$$

at $\theta = 0, \quad t = 0,$

at $\theta = \frac{\pi}{2}, \quad t = 1$ √

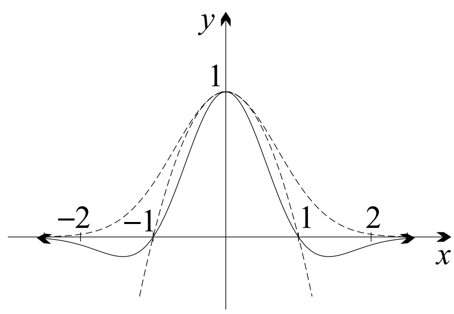
so $$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$ √

$$= \int_0^1 \frac{1}{(1+t)}\,dt$$ √

$$= \Big[\log(1+t)\Big]_0^1$$

$$= \log 2$$ √

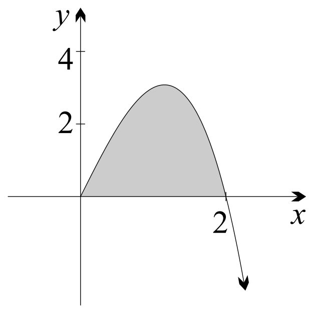
(d) **2**



Intercepts shown at $(0,1)$, $(-1,0)$ and $(1,0)$. √

Even and horizontal asymptote shown. √

(e) **3**



$$V = 2\pi \int_0^2 x \times y \, dx$$
$$= 2\pi \int_0^2 4x^2 - x^4 \, dx \quad \surd$$
$$= 2\pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \quad \surd$$
$$= 64\pi \left(\frac{1}{3} - \frac{1}{5} \right) - 0$$
so $V = \dfrac{128\pi}{15}$. √

Total for Question 11: 15 Marks

QUESTION TWELVE (15 marks) **Marks**

(a) **1**

For the hyperbola $5 = 4(e^2 - 1)$

so $e^2 = \frac{9}{4}$

thus $e = \frac{3}{2}$ √

(b) Differentiate $x^2 - 2xy + 3y^2 = 11$ implicitly to get **3**

$$2x - 2y - 2xy' + 6y'y = 0 \quad \surd$$

or $(3y - x)y' = y - x$

so $y' = \dfrac{y - x}{3y - x}$ √

and at $(2, -1)$ $y' = \frac{3}{5}$ √

(c) (i) Now $P'(z) = 4z^3 + 6z^2 + 6z + 4$, so **1**

$$P(-1) = 1 - 2 + 3 - 4 + 2$$
$$= 0$$

and $P(-1) = -4 + 6 - 6 + 4$
$$= 0 \quad \surd$$

Hence $z = -1$ is a double root.

(ii) Let one root be α **3**

then $\overline{\alpha}$ is also a root (real coefficients) √

product of roots: $\alpha \times \overline{\alpha} \times (-1) \times (-1) = 2$

so $|\alpha|^2 = 2$ ✓

sum of roots: $\alpha + \overline{\alpha} + (-1) + (-1) = -2$

so $\text{Re}(\alpha) = 0$

hence $\alpha = i\sqrt{2}$ ✓

and the roots are (-1), (-1), $i\sqrt{2}$ and $-i\sqrt{2}$.

(d) **3**

$$v\frac{dv}{dx} = -\tfrac{1}{3}v^3 - v$$

so $\dfrac{dx}{dv} = \dfrac{-3}{v^2+3}$

Integrate this result to get

$$x = -\tfrac{3}{\sqrt{3}}\tan^{-1}\tfrac{v}{\sqrt{3}} + C$$ ✓

At $t = 0$, $x = 0$ and $v = 3$ so

$$0 = -\sqrt{3}\tan^{-1}\sqrt{3} + C$$

or $C = \tfrac{\pi}{\sqrt{3}}$ ✓

So $x = \tfrac{\pi}{\sqrt{3}} - \sqrt{3}\tan^{-1}\tfrac{v}{\sqrt{3}}$

thus $\tan^{-1}\tfrac{v}{\sqrt{3}} = \tfrac{\pi}{3} - \tfrac{x}{\sqrt{3}}$

hence $v = \sqrt{3}\tan\left(\tfrac{\pi}{3} - \tfrac{x}{\sqrt{3}}\right)$ ✓

(e) (i) **2**

It should be clear that $r = \tfrac{3}{4}h$. ✓

Thus Area = square − corners + circle

$$= 6 \times 6 - 4 \times r^2 + \pi r^2$$

$$= 36 + (\pi - 4)r^2$$ ✓

$$= 36 + (\pi - 4)\tfrac{9}{16}h^2$$

(ii) **2**

Hence $V = \displaystyle\int_0^4 \left(36 + (\pi - 4)\tfrac{9}{16}h^2\right)\, dh$ ✓

$$= \left[36h + (\pi - 4)\tfrac{3}{16}h^3\right]_0^4$$

$$= 144 + 12(\pi - 4) - 0$$

$$= 12(8 + \pi)$$ ✓

Total for Question 12: 15 Marks

QUESTION THIRTEEN (15 marks) **Marks**

(a) (i) (α) In the following, write f for $f(x)$. **1**

Here $g(x) = \dfrac{1}{f}$

so $\dfrac{dg}{dx} = \dfrac{-1}{f^2} \times f'$ (by the chain rule)

Thus $g'(a) = -\dfrac{f'(a)}{\big(f(a)\big)^2}$

$= 0$ (since $f'(a) = 0$) ✓

(β) The product rule is easier, but most candidates used the quotient rule. **2**

$$g'' = -\frac{f^2 f'' - f' \times 2f' f}{f^4} \qquad \text{(by the quotient rule)}$$

$$= \frac{2(f')^2 - f'' f}{f^3}$$ ✓

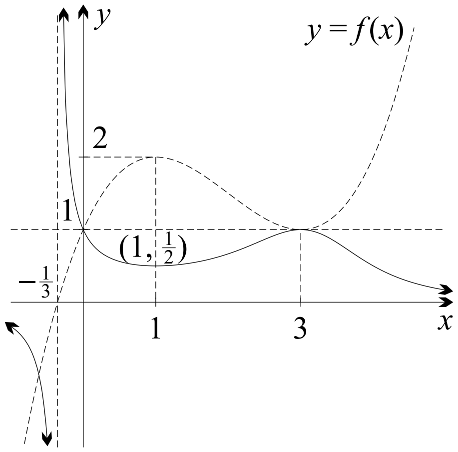
hence $g''(a) = \dfrac{0 - f''(a)f(a)}{\big(f(a)\big)^3}$

$= -\dfrac{f''(a)}{\big(f(a)\big)^2}$ ✓

The denominator is a square, so is positive,

hence $g''(a)$ has opposite sign to $f''(a)$.

(ii) **2**



Vertical and horizontal asymptotes ✓

Correct behaviour at turning points ✓

(b) (i) In $\triangle APT$ and $\triangle SPT$ **2**

$\angle APT = \angle SPT$ (reflection property of ellipse) ✓

$AP = PS$ (given)

$PT = PT$ (common)

thus $\triangle APT \equiv \triangle SPT$ (SAS) ✓

hence $AT = ST$ (matching sides congruent triangles)

(ii) **[1]**

$S'A = S'P + PA$

$= S'P + PS$ (by construction)

$= 2a$ (property of ellipse) ✓

(iii) (α) **[2]**

Likewise, $S'B = 2a$ and $BT = ST$ ✓

Hence in $\triangle S'AT$ and $\triangle S'BT$

$S'T = S'T$ (common)

$AT = BT = ST$ (proven)

$S'A = S'B = 2a$ (proven)

thus $\triangle S'AT \equiv \triangle S'BT$ (SSS) ✓

Hence $\angle AS'T = \angle BS'T$ (matching angles of congruent triangles.)

(c) (i) **[2]**

$\angle BCP = \angle BAP$ (angles in the same segment)

$= \alpha$ ✓

$\angle QCA = \angle QBA$ (angles in the same segment)

$= \beta$ ✓

Hence $\angle QCP = \alpha + \beta + \gamma$ (adjacent angles).

(ii) **[2]**

$2\alpha + 2\beta + \gamma = \pi$ (angle sum of $\triangle ABC$)

so $\alpha + \beta = \frac{1}{2}(\pi - \gamma)$ ✓

Thus $\angle QCP = \frac{1}{2}(\pi - \gamma) + \gamma$

$= \frac{1}{2}(\pi + \gamma)$

But γ is constant (angle in the major segment)

hence $\angle QCP = \frac{1}{2}(\pi + \gamma)$ is constant ✓

(iii) $\angle PCQ$ is constant hence **[1]**

QP is also constant (equal angles subtend equal chords) ✓

Total for Question 13: 15 Marks

QUESTION FOURTEEN (15 marks) **Marks**

(a) Using the given results: **3**

$$\begin{aligned}16\cos^3\theta\sin^2\theta &= 16\left(\frac{z+z^{-1}}{2}\right)^3\left(\frac{z-z^{-1}}{2i}\right)^2 \quad \surd\\ &= -\tfrac{1}{2}\left(z^3+3z+3z^{-1}+z^{-3}\right)\left(z^2-2+z^{-2}\right)\\ &= -\tfrac{1}{2}\left(z^5+z^3-2z-2z^{-1}+z^{-3}+z^{-5}\right) \quad \surd\\ &= -\tfrac{1}{2}\left(-2(z+z^{-1})+(z^3+z^{-3})+(z^5+z^{-5})\right)\\ &= -\tfrac{1}{2}(-2\cos\theta+2\cos 3\theta+2\cos 5\theta) \quad \surd\\ &= 2\cos\theta-\cos 3\theta-\cos 5\theta\,.\end{aligned}$$

(b) (i) By de Moivre, **1**

$$\operatorname{cis} 5\theta = \operatorname{cis} 2n\pi \qquad \text{where } z = \operatorname{cis}\theta$$

so $5\theta = 2n\pi$

or $\theta = \dfrac{2n\pi}{5}$.

Hence $z = 1, \operatorname{cis}\frac{2\pi}{5}, \overline{\operatorname{cis}\frac{2\pi}{5}}, \operatorname{cis}\frac{4\pi}{5}, \overline{\operatorname{cis}\frac{4\pi}{5}}$ $\surd$

(ii) **1**

$$\alpha^5 - 1 = 0$$

so $(\alpha-1)(\alpha^4+\alpha^3+\alpha^2+\alpha+1) = 0$ (GP theory)

hence $\alpha^4+\alpha^3+\alpha^2+\alpha+1 = 0$ (since $\alpha \neq 1$) $\surd$

(iii) **2**

Sum of roots: $\alpha^4+\alpha^3+\alpha^2+\alpha = -1$ (from part (ii)) $\surd$

Product of roots:
$$\begin{aligned}(\alpha^4+\alpha)(\alpha^3+\alpha^2) &= \alpha^7+\alpha^6+\alpha^4+\alpha^3\\ &= \alpha^2+\alpha+\alpha^4+\alpha^3 \quad (\text{since } \alpha^5 = 1)\\ &= -1 \quad (\text{by part (ii)})\end{aligned}$$

Hence a quadratic equation is $z^2+z-1=0$. $\surd$

(iv) **2**

$$\Delta = 5$$

so $z = \dfrac{-1\pm\sqrt{5}}{2}$ $\surd$

But α is in the first quadrant so take the positive solution.

Thus $\alpha^4+\alpha = 2\cos\frac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{2}$

hence $\cos\frac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$ $\surd$

(c) (i) Since $P(\alpha) = 0$ and $P(i\alpha) = 0$, α is a solution of **2**

$$z^4 - 2z^3 + 2z^2 - 10z + 25 = 0 \qquad [1]$$

and $$z^4 + 2iz^3 - 2z^2 - 10iz + 25 = 0 \qquad [2]$$ ✓

Subtracting [1] from [2], α is a solution of

$$(2i + 2)z^3 - 4z^2 - (10i - 10)z = 0$$

or $$2z\left((1+i)z^2 - 2z + 5(1-i)\right) = 0$$

thus $$(1+i)z^2 - 2z + 5(1-i) = 0 \quad (\text{since } P(0) \neq 0)$$ ✓

(ii) The discriminant is **4**

$$\Delta = 4 - 4 \times (1+i) \times 5(1-i)$$
$$= (6i)^2$$ ✓

so $$z = \frac{2 \pm 6i}{2(1+i)}$$ ✓
$$= \frac{1+3i}{1+i} \quad \text{or} \quad \frac{1-3i}{1+i}$$
$$= \frac{4+2i}{2} \quad \text{or} \quad \frac{-2-4i}{2}$$
$$= 2 + i \quad \text{or} \quad -1 - 2i$$ ✓

The other roots are conjugates ($P(z)$ has real coefficients)

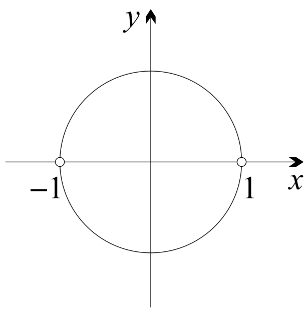
thus $z = 2+i, 2-i, -1+2i, -1-2i$ ✓

Total for Question 14: 15 Marks

QUESTION FIFTEEN (15 marks) **Marks**

(a) There are two obvious methods to do this question. Here is the first. **3**

$$\frac{z-1}{z+1} = \frac{(z-1)(\overline{z}+1)}{|z+1|^2}$$
$$= \frac{|z|^2 - 1 + z - \overline{z}}{|z+1|^2}$$
$$= \frac{|z|^2 - 1}{|z+1|^2} + \frac{2iy}{|z+1|^2} \qquad \text{(or equivalent)}$$ ✓



This is imaginary if $|z|^2 - 1 = 0$ ✓

provided that $y \neq 0$. ✓

That is, a circle centre the origin and radius 1, omitting the x-intercepts.

The other method is to note that $\arg\left(\dfrac{z-1}{z+1}\right) = \pm\dfrac{\pi}{2}$.

(b) (i) **1**

$$\begin{aligned}(Q'Q)^2 &= (a\mu - a\lambda)^2 + (b\mu - b\lambda)^2 \\ &= a^2(\mu-\lambda)^2 + b^2(\mu-\lambda)^2 \\ &= (\mu-\lambda)^2(a^2+b^2) \quad \surd\end{aligned}$$

so $Q'Q = (\mu - \lambda)\sqrt{a^2+b^2}$ (since $\mu > \lambda$).

(ii) In $\triangle AFP \parallel\!| \triangle AQR$ **1**

$$\frac{QR}{AR} = \frac{PF}{AF} \quad \text{(matching sides of similar triangles)}$$

so $\dfrac{b\mu}{a\mu - a} = \dfrac{b\tan\theta}{a\sec\theta - a}$

thus $\dfrac{\mu}{\mu - 1} = \dfrac{\tan\theta}{\sec\theta - 1}$ $\surd$

(iii) In $\triangle A'FP \parallel\!| \triangle A'Q'R'$ (AA) **1**

$$\frac{Q'R'}{A'R'} = \frac{PF}{A'F} \quad \text{(matching sides of similar triangles)}$$

so $\dfrac{b\lambda}{a\lambda + a} = \dfrac{b\tan\theta}{a\sec\theta + a}$

thus $\dfrac{\lambda}{\lambda + 1} = \dfrac{\tan\theta}{\sec\theta + 1}$ $\surd$

(iv) **2**

$$\begin{aligned}\frac{\mu}{\mu-1} \times \frac{\lambda}{\lambda+1} &= \frac{\tan\theta}{\sec\theta - 1} \times \frac{\tan\theta}{\sec\theta + 1} \\ &= \frac{\tan^2\theta}{\sec^2\theta - 1} \\ &= 1 \quad \surd\end{aligned}$$

thus $\lambda\mu = (\mu - 1)(\lambda + 1)$

or $\lambda\mu = \lambda\mu + \mu - \lambda - 1$

so $\mu - \lambda = 1$

hence $Q'Q = \sqrt{a^2+b^2}$ $\surd$

which is independent of the location of P on $\mathcal{H}$.

(c) Squares of real numbers are positive, so **1**

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

thus $x^2 + \dfrac{1}{x^2} \geq 2$

Now put $a = x^2$ to get the result

$$a + \frac{1}{a} \geq 2 \quad \surd$$

(d) (i) Rearrange the given equation to get **3**

$$\frac{dt}{dv} = \frac{-1}{g+kv}$$

thus $t = -\frac{1}{k}\log(g+kv) + C$ (for some constant C.) ✓

At $t=0$, $v = V_0$ so

$C = \frac{1}{k}\log(g+kV_0)$ ✓

Thus $t = \frac{1}{k}\log(g+kV_0) - \frac{1}{k}\log(g+kv)$

or $e^{kt} = \dfrac{g+kV_0}{g+kv}$

so $g+kv = g(1+\frac{k}{g}V_0)e^{-kt}$

hence $v(t) = \frac{g}{k}\Big((1+\frac{k}{g}V_0)e^{-kt} - 1\Big)$ ✓

(ii) At the maximum height the velocity is zero, so **1**

$(1+\frac{k}{g}V_0)e^{-kT} - 1 = 0$

or $1 = (1+\frac{k}{g}V_0)e^{-kT}$

hence $e^{kT} = (1+\frac{k}{g}V_0)$ ✓

(iii) **2**

$$\begin{aligned}
v(T-s)+v(T+S) &= \tfrac{g}{k}\Big((1+\tfrac{k}{g}V_0)e^{-kT+ks} - 1\Big) + \tfrac{g}{k}\Big((1+\tfrac{k}{g}V_0)e^{-kT-ks} - 1\Big) \\
&= \tfrac{g}{k}\Big((e^{kT}e^{-kT+ks} - 1) + (e^{kT}e^{-kT-ks} - 1)\Big) \quad \text{(part (ii))} \\
&= \tfrac{g}{k}(e^{ks} + e^{-ks} - 2) \\
&= \tfrac{g}{k}\left(a + \frac{1}{a} - 2\right) \quad \text{(where } a = e^{ks}\text{)} \\
&\geq \tfrac{g}{k}(2-2) \quad \text{(by part (c))} \\
&\geq 0\,.
\end{aligned}$$

✓ ✓

Total for Question 15: 15 Marks

QUESTION SIXTEEN (15 marks) **Marks**

(a) (i) The given equation will hold if **2**

$1 = A(x^2+1) + (Bx+C)(x+1)$

at $x = -1$ this gives

$1 = 2A$ so $A = \frac{1}{2}$ ✓

equating coefficients of x^2 gives

$0 = A + B$ so $B = -\frac{1}{2}$

and at $x = 0$ the result is

$1 = A + C$ so $C = \frac{1}{2}$ ✓

[Only full marks if all three correct.]

(ii) From the results of part (i), **3**

$$I(N) = \tfrac{1}{2}\int_0^N \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$= \tfrac{1}{2}\Big[\log(x+1) - \tfrac{1}{2}\log(x^2+1) + \tan^{-1}x\Big]_0^N$$ ✓✓

$$= \frac{1}{2}\Big(\log(N+1) - \tfrac{1}{2}\log(N^2+1) + \tan^{-1}N\Big)$$

$$= \tfrac{1}{4}\left(\log\left(\frac{(N+1)^2}{N^2+1}\right) + 2\tan^{-1}N\right)$$ ✓

(iii) Taking the limit of part (ii) as $N \to \infty$: **1**

$$\int_0^\infty \frac{dx}{x^3+x^2+x+1} = \lim_{N\to\infty} I(N)$$

$$= \lim_{N\to\infty} \tfrac{1}{4}\left(\log\left(\frac{(N+1)^2}{N^2+1}\right) + 2\tan^{-1}N\right)$$

$$= \lim_{N\to\infty} \tfrac{1}{4}\left(\log\left(\frac{(1+\frac{1}{N})^2}{1+\frac{1}{N^2}}\right) + 2\tan^{-1}N\right)$$

$$= \tfrac{1}{4}(\log 1 + 2\times\tfrac{\pi}{2})$$

$$= \tfrac{\pi}{4}$$ ✓

(b) (i) Using integration by parts $\int v'u\,dx = uv - \int u'v\,dx$, **1**

put $u = e^{-x}$ and $v' = x^n$

so $u' = -e^{-x}$ and $v = \frac{1}{n+1}x^{n+1}$

then $$I_n = \left[\frac{x^{n+1}e^{-x}}{n+1}\right]_0^1 + \int_0^1 \frac{x^{n+1}e^{-x}}{n+1}\,dx$$

$$= \frac{e^{-1}-0}{n+1} + \frac{1}{n+1}\int_0^1 x^{n+1}e^{-x}\,dx$$ ✓

$$= \frac{1}{n+1}\Big(e^{-1} + I_{n+1}\Big)$$

(ii) (A) When $n = 1$ **3**

$$\text{LHS} = \frac{1}{1}\left(e^{-1} + I_1\right) \qquad \text{(by part (i))}$$

$$= e^{-1}\times\frac{1}{1!} + \frac{1}{1!}\times I_1$$ ✓

$$= \text{RHS}$$

so the result is true for $n = 1$.

(B) Assume the result is true when $n = k$. That is, assume that

$$I_0 = e^{-1}\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{k!}\right) + \frac{1}{k!}\,I_k \qquad (\dagger\dagger)$$

Now prove the result is true when $n = k+1$. That is, prove that

$$I_0 = e^{-1}\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{k!} + \frac{1}{(k+1)!}\right) + \frac{1}{(k+1)!}\,I_{k+1}$$

Now, by part (i), the result in (††) becomes

$$\begin{aligned} I_0 &= e^{-1}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!}\right)+\frac{1}{k!}\times\frac{1}{k+1}\left(e^{-1}+I_{k+1}\right) \quad \boxed{\surd}\\ &= e^{-1}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!}\right)+\frac{e^{-1}}{(k+1)!}+\frac{1}{(k+1)!}\,I_{k+1}\\ &= e^{-1}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!}+\frac{1}{(k+1)!}\right)+\frac{1}{(k+1)!}\,I_{k+1} \quad \boxed{\surd}\end{aligned}$$

as required.

It follows from parts (A) and (B) by mathematical induction that the result is true for all positive integers n.

(iii) Take the limit as $n \to \infty$ as indicated to get **[1]**

$$I_0 = e^{-1}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\right)$$

but

$$\begin{aligned} I_0 &= \int_0^1 e^{-x}\,dx\\ &= \Big[e^{-x}\Big]_0^1\\ &= -e^{-1}+1\end{aligned}$$

Hence $1-e^{-1} = e^{-1}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\right)$ $\boxed{\surd}$

so $\qquad e-1 = \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$

or $\qquad e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$

(iv) Let $e = \dfrac{p}{q}$ then **[1]**

$$\frac{p}{q} = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{q!}+\frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\ldots$$

Now multiply both sides by $q!$ to get

$$\frac{p\,q!}{q} = \underbrace{q!+\frac{q!}{1!}+\frac{q!}{2!}+\frac{q!}{3!}+\ldots+\frac{q!}{q!}}+\frac{q!}{(q+1)!}+\frac{q!}{(q+2)!}+\ldots \quad \boxed{\surd}$$

or $\quad p(q-1)! = \left(\displaystyle\sum_{k=0}^{q}\frac{q!}{k!}\right) + \dfrac{1}{q+1}+\dfrac{1}{(q+1)(q+2)}+\ldots$

(v) Let $f = \dfrac{1}{q+1} + \dfrac{1}{(q+1)(q+2)} + \dfrac{1}{(q+1)(q+2)(q+3)} + \ldots$ then **2**

$$f < \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \ldots \qquad \text{(decrease denominators)} \quad \surd$$

$$< \frac{1}{q+1} \times \frac{1}{1 - \frac{1}{q+1}} \qquad \text{(infinite sum of a GP)}$$

$$< \frac{1}{q}$$

$$< \frac{1}{2} \qquad \text{(since } q > 2\text{)} \quad \surd$$

Also $f > 0$ since every term is positive. Hence $0 < f < \frac{1}{2}$.

(vi) The LHS of part (iv) is an integer. **1**
The RHS of part (iv) is an integer plus a fraction.

This is a contradiction and hence e is a irrational. $\surd$

Total for Question 16: 15 Marks

DNW